of geometry.[1]   This and other instances of such ignorance are set forth by J. S. Mackay in "Herbert Spencer and mathematics," *Proceedings of the Edinburgh Mathematical Society*, 1907, volume 25, pp. 95–106. ARC.

## PROBLEMS—SOLUTIONS

**2800 [1920, 31].   Proposed by A. M. HARDING, University of Arkansas.**

If $x + y + z = xyz$, show that

$$\frac{2x}{1 - x^2} + \frac{2y}{1 - y^2} + \frac{2z}{1 - z^2} = \frac{2x}{1 - x^2} \cdot \frac{2y}{1 - y^2} \cdot \frac{2z}{1 - z^2}.$$

### SOLUTION BY ARNOLD DRESDEN, University of Wisconsin.

Except for values of $x$ and $y$ for which $xy = 1$, the relation (1) $x + y + z = xyz$ is equivalent to $z = (x + y)/(xy - 1)$.   Direct substitution of this value for $z$ shows that the formula

(2)
$$\frac{2x}{1 - x^2} + \frac{2y}{1 - y^2} + \frac{2z}{1 - z^2} = \frac{2x}{1 - x^2} \cdot \frac{2y}{1 - y^2} \cdot \frac{2z}{1 - z^2}$$

holds for all values of $x$, $y$, except $x = \pm 1$; $y = \pm 1$; $xy = 1$, for which values one or the other of these equations loses meaning.

It is readily seen that an infinite number of other formulas will hold true as a consequence of (1), viz.,

(3)         $\Sigma \tan (n \tan^{-1} x) = \Pi \tan (n \tan^{-1} x)$, for $n = 1, 2, \cdots$,

the sum and product to be extended over $x$, $y$, and $z$.   For since $x + y + z = xyz$, it follows that $- \tan^{-1} x - \tan^{-1} y = \tan^{-1} z$, for some determination of $\tan^{-1} z$; in virtue of this relation we have then $- n \tan^{-1} x - n \tan^{-1} y = n \tan^{-1} z$ and therefore:

$$- \tan (n \tan^{-1} z) = \tan (n \tan^{-1} x + n \tan^{-1} y) = \frac{\tan (n \tan^{-1} x) + \tan (n \tan^{-1} y)}{1 - \tan (n \tan^{-1} x) \cdot \tan (n \tan^{-1} y)} ;$$

*i.e.*, $\Sigma \tan (n \tan^{-1} x) = \Pi \tan (n \tan^{-1} x)$.

It would be interesting to know the general solution of the functional equation:

$$\sum_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} f(x_i),$$

under the restriction $\Sigma x_i = \Pi x_i$.

Also solved by NORMAN ANNING, T. M. BLAKSLEE, S. M. BERG, J. A. BULLARD, H. N. CARLETON, E. S. HAMMOND, C. N. MILLS, E. J. OGLESBY, H. L. OLSON, J. L. RILEY, ARTHUR PELLETIER, H. S. UHLER, W. R. WARNE, and C. C. WYLIE.

**2802 [1920, 31].   Proposed by WARREN WEAVER, California Institute of Technology.**

Consider two circles, each of radius $k$, with centers at $(0, 0)$ and $(k', 0)$ respectively, where $k'$ is less than $k$.   Through the point $(k', 0)$ draw a ray making an angle $\theta$ with the positive $x$-axis. Call the intersection of this line with the first circle $A$, and with the second circle $B$.   Extend the line through the point $(k', 0)$ in the opposite direction, and call the intersection of this extension with the first circle $A'$, and with the second circle $B'$.   Prove that the sum of the two segments $AB$ and $A'B'$ is independent of $k$, and depends only upon $k'$, i.e. the shift of the circles, and $\theta$.

### SOLUTION BY U. G. MITCHELL, University of Kansas.

Let the points $A$, $B$, $A'$, $B'$ have coördinates $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$, respectively. By the conditions of the problem the points $A = (x_1, y_1)$ and $A' = (x_3, y_3)$ are on the circle $x^2 + y^2 = k^2$, the points $B = (x_2, y_2)$ and $B' = (x_4, y_4)$ are on the circle $(x - k')^2 + y^2 = k^2$, and all four of the points $A$, $B$, $A'$, $B'$ are on the line $y = m(x - k')$, where $m = \tan \theta$.

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[1] His boasting in the preface to the American edition, in 1876, of *Inventional Geometry: a series of problems*, by his father W. G. Spencer, will be recalled.

Since for a given circle and a given exterior point the product of any secant through the point and the exterior segment of the secant is equal to the square of the tangent from the point to the circle we have $A'B \cdot AB = x_2^2 + y_2^2 - k^2$ and $A'B \cdot A'B' = (x_3 - k')^2 + y_3^2 - k^2$.  Hence,

$$AB + A'B' = \frac{x_2^2 + y_2^2 - k^2 + (x_3 - k')^2 + y_3^2 - k^2}{A'B}.$$

By use of $A'B = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}$ and the relations given above this equation readily reduces to $AB + A'B' = 2k'/\sqrt{1 + m^2}$, which is independent of $k$ and dependent upon $k'$ and $\theta$ only, as was to be shown.

Also solved by T. M. BLAKSLEE, PAUL CAPRON, OLIVE C. HAZLETT, C. N. MILLS, E. J. OGLESBY, ARTHUR PELLETIER, C. P. SOUSLEY, C. E. STROMQUIST, and H. S. UHLER.

**2804 [1920, 32].  Proposed by T. H. GRONWALL, Washington, D. C.**

Show that for $|x| < 1$

$$\frac{1}{\sqrt{1 - x^4}} \int_0^x \frac{dx}{\sqrt{1 - x^4}} = x + \sum_1^\infty \frac{3 \cdot 7 \cdots (4n - 5)(4n - 1)}{5 \cdot 9 \cdots (4n - 3)(4n + 1)} x^{4n+1},$$

$$\left( \int_0^x \frac{dx}{\sqrt{1 - x^4}} \right)^2 = x^2 + \sum_1^\infty \frac{3 \cdot 7 \cdots (4n - 5)(4n - 1)}{5 \cdot 9 \cdots (4n - 3)(4n + 1)} \frac{x^{4n+2}}{2n + 1}.$$

### SOLUTION BY O. S. ADAMS, Coast and Geodetic Survey.

The general forms for such integrals may be derived and the formulas in the statement will then be special cases of the derived forms.  We then state the problem as follows:

For $|x| < 1$, $s$ a positive integer,

$$\frac{1}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}} = x + \sum_1^\infty \frac{\dfrac{s + 2}{2} \cdot \dfrac{3s + 2}{2} \cdots \dfrac{(2n - 1)s + 2}{2}}{(s + 1)(2s + 1) \cdots (ns + 1)} x^{ns+1},$$

$$\left( \int_0^x \frac{dx}{\sqrt{1 - x^s}} \right)^2 = x^2 + \sum_1^\infty \frac{\dfrac{s + 2}{2} \cdot \dfrac{3s + 2}{2} \cdots \dfrac{(2n - 1)s + 2}{2}}{(s + 1)(2s + 1) \cdots (ns + 1)} \frac{2x^{ns+2}}{ns + 2}.$$

Let

$$\phi(x) = \int_0^x \frac{dx}{\sqrt{1 - x^s}};$$

then

$$\frac{d}{dx} \phi(x) = \frac{1}{\sqrt{1 - x^s}},$$

and

$$2\phi(x) \frac{d}{dx} \phi(x) = \frac{2}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}}.$$

By integration,

$$\phi^2(x) = 2 \int_0^x \left[ \frac{1}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}} \right] dx.$$

Therefore, if the first series is multiplied by $2dx$ and integrated from $0$ to $x$, the second series will be obtained.

Let

$$y = \frac{1}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}};$$

then

$$\frac{dy}{dx} = \frac{1}{1 - x^s} + \frac{(s/2)x^{s-1}}{(1 - x^s)^{3/2}} \int_0^x \frac{dx}{\sqrt{1 - x^s}} = \frac{1}{1 - x^s} + \frac{(s/2)x^{s-1}y}{1 - x^s};$$